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Integral geometry on families of surfaces in the space

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Dedicated to Israel Moiseevich Gelfand on the occasion of his 75th birthday

Abstract. *A complex $\{B_\xi\}$ of submanifolds is called admissible in the sense of integral geometry if there are such densities μ_ξ on B_ξ, that the integral transformation*

$$I : f(x) \to \int_{B_\xi} f(x)\, \mu_\xi$$

has a local inversion formula.
We prove that compact smooth surfaces of an admissible complex in P^3 has degree $\leqslant 3$ and obtain complete classification of admissible complexes of quadrics. Several general theorems and conjectures about admissible complex of k-dimensional submanifolds are stated.

§ 1. INTRODUCTION

1. **Basic definitions.** n-parametric family of submanifolds B_ξ of n-dimensional manifold B is called admissible, if the value of any function f on B at every point x can be recovered from integrals of f over submanifolds, passing through an infinitesimal neighbourhood of x.

The general conception of integral geometry has been proposed by I.M. Gelfand in the end of 50s [Gel].

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Let us give rigorous definitions. A double fibration is a diagram of manifolds

$$(1) \qquad \begin{array}{ccc} & A & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ B & & \Gamma \end{array}$$

such that $\pi_1 \times \pi_2 : A \to B \times \Gamma$ is an embedding. For $x \in B$ and $\xi \in \Gamma$ set

$$B_\xi = \pi_1 \circ \pi_2^{-1}\xi; \quad \Gamma_x = \pi_2 \circ \pi_1^{-1}x$$

A double fibration axiomatises notion of a family of submanifolds. To define diagram (1) it is sufficient to set the family of submanifolds B_ξ in B, parametrised by Γ, or the family Γ_x in Γ (where $x \in B$). The last one is called dual to the initial one.

Let us suppose also that manifolds B_ξ are nonsingular.

Choose some densities μ_ξ on manifolds B_ξ and define the operator of integration

$$I : f(x) \to \int_{B_\xi} f(x)\, \mu_\xi$$

In other words I is an operator from $C_0^\infty(B)$ to $C^\infty(\Gamma)$ whose Schwartz kernel is $\mu(x, \xi) \cdot \delta(A) \cdot d\overline{\overline{B}}$, where $\delta(A)$ is a δ-function of the submanifold A, db is a volume element on B and $\mu(x, \xi)$ is a function on $B \times \Gamma$.

If there is an inverse operator $J : C^\infty(\Gamma) \to C^\infty(B)$ whose Schwartz kernel has the form $L \cdot \delta(A) \cdot d\gamma$ where L is a differential operator on $B \times \Gamma$ and $d\gamma$ is a volume element on Γ, we say that the integral operator I has a local inversion formula. It is clear, that in this case $\dim B \leqslant \dim \Gamma$.

From now on we will assume that all manifolds are *complex algebraic*, while we will integrate smooth functions. The reason is that the study of the integral transform I in the complex case is much more simpler than in the real one. For example, if $\dim B_\xi$ is odd, then there are no local inversion formulas.

Let us suppose that $\dim B = \dim \Gamma$. Then a double fibration in the category of complex manifolds is called *admissible* if there are such $(n, 0)$ – forms μ_ξ that for the integral transform

$$I : C_0^\infty(B) \to C^\infty(\Gamma); \; I : f(x) \to \int_{B_\xi} f(x)\, \mu_\xi\, \bar{\mu}_\xi$$

there exists an inverse operator J with the Schwartz kernel of the form $L \cdot \bar{L} \cdot \delta(A) \cdot d\gamma\, d\bar{\gamma}$ where L is a holomorphic differential operator on $B \times \Gamma$.

In this case densities $\mu_\xi \cdot \bar{\mu}_\xi$ are called admissible densities (for the double fibration (1)).

2. **Necessary condition for admissibility** [Gon 2]. Let $T^*_Y X$ be the conormal bundle to a submanifold $Y \subset X$.

Consider the following simplectisation of diagram (1):

$$\begin{array}{ccccc} & & T^*_A(B \times \Gamma) \hookrightarrow T^*(B \times \Gamma) & & \\ & \swarrow^{\rho_B} & & \searrow^{\rho_\Gamma} & \\ T^*B & & & & T^*\Gamma \end{array} \tag{2}$$

where ρ_B and ρ_Γ are the restrictions to $T^*_A(B \times \Gamma)$ of the projections from $T^*(B \times \Gamma) = T^*B \times T^*\Gamma$ on factors.

In the case $\dim B = \dim \Gamma$ all manifolds in diagram (2) are of equal dimensions. We denote by $d(B)$ (respectively $d(\Gamma)$) the degree of the map $\rho_B(\rho_\Gamma)$.

If $\dim B = \dim \Gamma$, then traditionally the family of submanifolds B_ξ is called a *complex.*

Example. Let Γ be a complex of lines in $\mathbb{C}P^3$. Then classics defined its degree as follows. Consider the embedding of the Grassmanian of lines in $\mathbb{C}P^3$ as a Plukker quadric Q_4 in $\mathbb{C}P^5$. Then $\Gamma = Q_4 \cap H$, where H is a hypersurface in $\mathbb{C}P^5$. By definition, degree of Γ is equal to degree of H.

It can be proved that $d(B)$ coincides with this degree. So we will say that $d(B)$ is the *degree* and $d(\Gamma)$ is the *codegree* of the double fibration (1).

THEOREM 1.1. [Gon 1-2]. *If* $\dim B = \dim \Gamma$ *and the double fibration (1) is admissible, then* $d(\Gamma) = 1$. ■

CONJECTURE A. *Every complex of k-dimensional planes in* $\mathbb{C}P^n$ *of codegree 1 is admissible.*

This conjecture is verified for linear complexes of k-dimensional planes [M] & [Gon1] and for line complexes (it follows from results of I.M. Gelfand and M.I. Graev [G Gr 1].

In [Gon 3] we will prove it for complexes of $(n-2)$-planes in $\mathbb{C}P^n$ and 2-planes in $\mathbb{C}P^5$.

V. Guillemin and S. Sternberg proved (see [GS1-2]) that if in the C^∞-cathegory the map

$$\rho^0_\Gamma : T^*_A(B \times \Gamma) \setminus 0 \to T^*\Gamma \setminus 0$$

is injective immersion, $\pi_1 : A \to B$ is proper and I^t is the operator of integration over Γ_x (equipped with a measure), then $I^t \circ I$ is an elliptic pseudodifferential operator.

But in the complex case there are no such examples if codim $B_\xi > 1$. (see [Gon 2]).

3. Admissible families of curves. The full description of admissible double fibrations in the case dim $B_\xi = 1$ was obtained in the end of 70-s ([GGiSh], [GGr2], [BG1-2], [B], [Gi 1-2]). It was done in two steps. First of all, I.M. Gelfand, S.G. Gindikin, Z. Ya. Shapiro and M.I. Graev had founded necessary and sufficient conditions for the admissibility of a double fibration in the case dim $B = \dim \Gamma$, $\dim B_\xi = 1$. ([GGiSh], [GGr2]).

In particularly, they proved that B_ξ is a rational curve (for admissible family of curves).

The next step has been performed by J.N. Bernstein and S.G. Gindikin ([BG 1-2], [B], [Gi 1-2]).

Let us state a part of their results: the description of admissible families of curves in generic position.

THEOREM 1.2. ([BG 1-2]). *Let* Γ' *be a complete family of compact nonsingular rational curves on B (i.e.* $\dim \Gamma' = H^0\ (B_\xi, N_{B_\xi}, B)$*). Then the family of curves, which are tangent to* r_1 *fixed hypersurfaces and intersect* r_2 *fixed subvarieties of codimension 2 in B, where* $r_1 + r_2 \leqslant \dim \Gamma - \dim B$ *is admissible and all admissible families of curves in generic position can be obtained by this way.* ∎

For example, any admissible complex of lines in $\mathbb{C}P^3$ consists of lines, which are either tangent to an algebraic surface or interset an algebraic curve.

For admissible complexes of lines theorem 1.2 was proved by I.M. Gelfand and M.I. Graev [GGr 1]).

Other admissible complexes in Γ' can be obtained from these by a limit procedure. It is remarkable, that there is an explicit construction for all admissible complexes in Γ'. We will give it in s. 2 of §3. (See theorem 3.9).

DEFINITION 1.3. *The variety of critical values of the projection* $\pi_1 : A \to B$ *is called the critical variety for a family* $\{B_\xi\}$ *on B.* ∎

For example, the critical variety for admissible complexes in generic position in Γ' is the union of r hypersurfaces and r subvarieties of codimension 2, mentioned in the formulation of theorem 1.2.

Theorem 1.2 tells us, that in the case of curves admissible complexes *are characterised* by the property, that they can be defined by tangency – intersection conditions with the critical variety.

The situation in the case dim $B_\xi \geqslant 2$ is much more complicated. For example, a complex of k-planes, which are tangent to $(k + 1)(n - k) - n$ hypersurfaces in CP^n has codegree

$$C_n^k - ((k + 1)(n - k) - n)$$

which is greater than 1 if $k \neq 1, n - 2$. where $((k + 1)(n - k)$ is the dimension of the Grassmanian of k-planes).

4. Main results. As we see in the previous section, there is the exhaustive description of admissible families of curves.

Unlike this, almost nothing was known about manydimensional case. For example, the only known admissible families of surfaces in 3-dimensions was

a) complex of all planes in R^3 of C^3.

b) complex of spheres, tangent to a plane.

The first is the classical Radon transformation, and the second is the horospheric transformation in the Lobatchevsky space, because horospheres in upper-halfplane realisation are exactly spheres, which are tangent to absolute (i.e. the plane $z = 0$) (see [GGV]).

In §§3, 4 of this article we classify all admissible complexes of surfaces of the second order (i.e. quadrics) in CP^3. First of all we give in §3 some examples of such complexes in CP^3 and prove that they are admissible. Then we prove in §4 that there are no others complexes of quadrics in CP^3 of codegreee 1. So we proved that any complex of quadrics in CP^3 of codegree 1 is admissible. This result confirms our general conjecture.

CONJECTURE B. *If* Γ *is an admissible family of submanifolds* B_ξ *in* B *and* $\dim B < \dim \Gamma$, *then every complex in* Γ *of codegree 1 is admissible.* ■

The definition of admissibility in the case $\dim B < \dim \Gamma$ is based on the notion of *universal* local inversion formula, discovered for k-planes in CP^n by I.M. Gelfand, M.I. Graev and Z. Ya. Shapiro in their beautiful paper [GGSh1] more than 20 years ago. We will give it in §2.

What can we say about admissible families of surfaces in CP^3? There is the following rationality theorem.

THEOREM 1.4. [Gon2] *If* B_ξ *is a complex of submanifolds in* B *of codegree 1, then* B_ξ *is rational and there is the canonical rational structure on* $PT^*_{B_\xi}B$. ■

Recall, that by Chow theorem any compact analytic subvariety in $\mathbb{C}P^n$ is algebraic. So if $\{B_\xi\}$ is an admissible complex of compact smooth surfaces in $\mathbb{C}P^3$, then B_ξ is a surface of degree $\leqslant 3$, because it is known, that there are no rational smooth surfaces in P^3 of degree greater than 3 (see [GH] ch. IV).

We hope to classify admissible complexes of cubic surfaces in $\mathbb{C}P^3$ in the subsequent paper.

In most cases admissible complex of quadrics in $\mathbb{C}P^3$ can be reduced to an admissible complex of curves on a 3-dimensional manifold in a following manner: there is a diagram

$$
(4) \qquad \begin{tikzcd}[column sep=small] & & A \arrow[dl] \arrow[dr] & & \\ & A_1 \arrow[dl] \arrow[dr] & & A_2 \arrow[dl] \arrow[dr] & \\ B & & L & & \Gamma \end{tikzcd}
$$

where $B \cong \mathbb{C}P^3$, $B \leftarrow A_1 \rightarrow L_1$ is an admissible complex of lines in $\mathbb{C}P^3$, $L \leftarrow A_2 \rightarrow \Gamma$ is an admissible complex of curves on L and $\mathbb{C}P^3 \leftarrow A \rightarrow \Gamma$ is the admissible complex of quadrics. So the integral transformation, connected with it is the composition

$$C_0^\infty(\mathbb{C}P^3) \xrightarrow{I_1} C^\infty(L) \xrightarrow{I_2} C^\infty(\Gamma)$$

Thus the admissibility of this complex of quadrics follows immediately from admissibility of double fibrations in the lower part of the diagram (4).

In [Gon3] we will prove that reduction to curves is not a lucky chance, but a rule for admissible complex in the case $\dim B_\xi > 1$. See also §5.

§2. UNIVERSAL LOCAL INVERSION FORMULA AND ADMISSIBILITY IN THE CASE $\dim B < \dim \Gamma$

Suppose that $\dim B < \dim \Gamma$. Then $\dim B_\xi < \dim \Gamma_x$. Let $\varkappa_x : C^\infty(\Gamma) \rightarrow \Omega^k(\Gamma_x)$ be a differential operator of order $k = \dim B_\xi$ such that $d\varkappa_x(If) = 0$ for any $f \in C_0^\infty(B)$.

If γ is a K-dimensional cycle in Γ_x, then

$$\int_\gamma \varkappa_x(If) = c(\gamma) \cdot f(x)$$

where $c(\gamma)$ does not depend on f. Indeed, the integral depends only on the homology class of γ, so it defines a generalised function on B which is concentrated in the point x (because $\dim \Gamma_x > \dim \gamma$). The homogeneity considerations shows that it is proportional to $\delta(x)$.

If for any point x there are a cycle γ in Γ_x and an operator $\varkappa_x$ such that $c(\gamma) \neq 0$, then we will say that the integral transform I admits an *universal local inversion formula*. Various inversion formulas can be obtained choosing appropriate cycles $\tilde{\gamma}$, homological to γ.

As usually, in the complex case we suppose that the integral has a form

$$\int_\gamma \varkappa_x \wedge \bar{\varkappa}_x\,(If)$$

DEFINITION 1.2. *We suppose that* dim $B <$ dim Γ. *Then the double fibration* (1) *is said to be admissible, if there are such measures of type* $\chi_\xi \bar{\chi}_\xi$ *on* B_ξ *that the corresponding integral transform* I *admits an universal local inversion formula.* ■

Let h be a generic hyperplane in $T_\xi \Gamma$. We denote by C_x the homology class of codimension $2k$ in Γ_x represented by the cycle, consisting of all such points $\zeta \in \Gamma_x$, that $T_x B_\zeta \subset h$.

PROPOSITION 2.2. *In the complex case* $c(\gamma)$ *is equal to* $\langle c_x\ \gamma \rangle$ *where* $\langle\ ,\ \rangle$ *is the intersection form in* $H_{2k}(\Gamma_x)$. ■

§3. ADMISSIBLE COMPLEXES OF QUADRICS IN THE THREE–DIMENSIONAL SPACE; EXAMPLES

1. We start with examples

Example 3.1. Complex of quadrics, containing a fixed line ℓ and tangent to 3 subvarieties M_1, M_2, M_3 in $\mathbb{C}P^3$.

Example 3.2. a). Complex, consisting of quadrics, which are tangent to a fixed quadric Q along a plane conic and some more a subvariety in $\mathbb{C}P^3$.

Example 3.2.b). Complex of quadrics, passing through a fixed plane conic C and tangent to a subvariety in $\mathbb{C}P^3$.

Complex from example 3.2.b) is projectively equivalent to complex of spheres.

If we set the quadric Q to the absolute, then quadrics in RP^3 from example 3.2.a) are spheres in the Kely-Kleine realisation of non-equilidian geometry: (the distance between any two points A, B in RP^3 is given by the formula

$$(A, B) = \frac{1}{2i} \log (A, B, P_1, P_2)$$

where P_1 and P_2 are points of intersection of the line AB with the absolute and $(\cdot\,,\cdot\,,\cdot\,,\cdot)$ is the cross ratio.

Consider the sphere $S^3 = \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$.

Then the stereographic projection from the point (1, 0, 0, 0) transforms spheres in the hyperplane $x_0 = -1$ to hyperplane sections of S^3, which are also spheres. Further, the projection from the origin onto the hyperplane $x_0 = -1$, which is 2 : 1 map, transform spheres on S^3 to spheres in Kely-Klein realisation of non-euclidean geometry with the quadric $x_1^2 + \ldots + x_3^2 = 1$ as the absolute. So the composition of this two transformations provides the 2 : 1 covering, which transfers the euclidian spheres to the non-euclidian.

Example 3.3. Let R be a ruled (but not a developable!) surface in $\mathbb{C}P^3$.

Consider complex, consisting of quadrics, which are tangent to R along rectilinear generatrices of R and some more a given subvariety M.

Note, that the tangent plane in $\mathbb{C}P^3$ to a developable surface does not change along a rectilinear generatrix. so quadric cannot be tangent to developable surface along a line.

Tangent planes to ruled surface R along a rectilinear generatrix ℓ gives a $1:1$ map from $\ell \cong \mathbb{C}P^1$ to a manifold of all planes, containing ℓ (which is also $\mathbb{C}P^1$). So there is exactly 3-parametric family of quadric, which are tangent to R along ℓ.

Example 3.4. Complex, consisting of quadrics, which are tangent to a given plane H at a fixed point x and some more 3 subvarieties in $\mathbb{C}P^3$.

If π is the infinite plane in $\mathbb{C}P^3$, then

$$x_3 = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + a_1x_1 + a_2x_2 + a_0$$

is the equation of quadrics, which are tangent to π at the point $x = (0:0:0:1)$.

2. THEOREM 3.5. *All complexes from examples* 3.1-3.3 *are admissible.*

Proof. First of all let us verify that any of these complexes is a composition of admissible complexes of curves, i.e. it has the form (4).

Denote by L_ℓ(respectively L_Q, L_C and L_R) the manifold of all lines, intersecting the line ℓ (respectively tangent to the quadric Q, along a plane conic passing through the plane conic C or tangent to ruled surface R). Then L_ℓ, L_Q, L_C and L_R are admissible complexes of lines (see theorem 1.2).

LEMMA 3.6. *If we set* $L = L_\ell$ *(respectively* L_Q, L_C *of* L_R*), we can represent a complex from example 3.1 (respectively 3.2 or 3.3) in the form (4).*

Proof. If B_ξ is a quadric containing the line ℓ (tangent to R along the line or to Q along a conic) then all lines on B_ξ, intersecting this line, form a rational curve on L_ℓ (L_R or L_Q). We denote it by $\breve{B}_\xi$. If B_ξ is a quadric, containing the plane conic C, then all lines on it form a pair of rational curves on L_C (we also denote it by $\breve{B}_\xi$).

Let us prove that in any case we obtain the complete family of rational curves $\breve{B}_\xi$ on L. Indeed, every variation $\breve{B}_\xi(t)$ of the curve $\breve{B}_\xi$ provides the variation $B_\xi(t)$ of the quadric B_ξ. Then $B_\xi(t)$ is a quadric, because the quadric can not be deformed. It follows from the definition of L that $B_\xi(t)$ contain the line ℓ (respectively intersect the conic C or tangent R along the line).

Now let us prove that complex of rational curves, corresponding to the complex of quadrics from example 3.1 is admissible. (The proof of the same statement for complexes from examples 3.2 and 3.3 is in complete analogy with this one). Let $\breve{M}_i$ $(i = 1, 2, 3)$ be the surface in L, consisting of lines, tangent to M_i. Then it is not hard to see that a complex from example 3.1 corresponds to the complex of curves $\check{B}_\xi$, which are tangent to surfaces $\check{M}_1, \check{M}_2, \check{M}_3$. Therefore it is admissible by theorem 1.2. ∎

Remark 3.7. The codegree of the double fibration $B \leftarrow A \rightarrow \Gamma$ in (4) is the product of codegrees of double fibrations in the lower part of (4). So if the codegree of the double fibration $B \leftarrow A \rightarrow \Gamma$ is 1, then the same is right for lower double fibrations in (4). Therefore the double fibration $L \leftarrow A_2 \rightarrow \Gamma$ is admissible (because $\check{B}_\xi$ are curves) and if $\dim B = 3$, then the double fibration $B \leftarrow A_1 \rightarrow L_1$ is also admissible (by the same reason).

LEMMA 3.8. *If double fibrations in the lower part of diagram (4) are admissible, then their composition – the double fibration $B \leftarrow A \rightarrow \Gamma$ – is also admissible.*

Proof. We can choose some admissible densities χ_ξ on curves $\check{B}_\xi$. Let η be a point of L, $\ell(\eta)$ -corresponding line in $\mathbb{C}P^3$ and ψ_η -admissible density on the line $\ell(\eta)$ (Recall, that the admissibility of these densities means, that the corresponding integral transformation has a local invertion formula – see s.1 of §1).

We define the integral transformation $I : C_0^\infty(\mathbb{C}P^3) \rightarrow C^\infty(\Gamma)$ as follows:

$$I : f(x) \mapsto \int_{\check{B}_\xi} \left(\int_{\ell(\eta)} f(x)\,\psi_\eta \right) \chi_\xi$$

So, we obtain:

$$C_0^\infty(\mathbb{C}P^3) \xrightarrow{I_{\psi_\eta}} C^\infty(L) \xrightarrow{I_{\chi_\xi}} C^\infty(\Gamma); \; I = I_{\chi_\xi} \circ I_{\psi_\eta}$$

where

$$I_{\psi_\eta} : f(x) \mapsto \varphi(\eta) := \int_{\ell(\eta)} f(x)\ \psi_\eta$$

$$I_{\chi_\xi} : \varphi(\eta) \mapsto \int_{\check{B}_\xi} \varphi(\eta)\,\chi_\xi$$

Denote by J_{ψ_η} and J_{χ_ξ} inverse operators for I_{ψ_η} and I_{χ_ξ} which are local inverse formula for these integral transformations. Then $J = J_{\psi_\eta} \circ J_{\chi_\xi}$ is the local inverse operator for the integral transformation I.

Recall, that in the case of example 3.2 a quadric B_ξ, passing through the conic C, transfers to the union of two rational curves: $\breve{B}_\xi^{(1)}$ and $\breve{B}_\xi^{(2)}$, which are parametrised two families of lines on B_ξ. Thus we must choose such admissible densities $\chi_\xi^{(i)}$ on $\breve{B}_\xi^{(i)}$ that

$$\int_{\breve{B}_\xi^{(1)}} \left(\int_{\ell(\eta)} f(x)\,\psi_\eta \right) \chi_\xi^{(1)} = \int_{\breve{B}_\xi^{(2)}} \left(\int_{\ell(\eta)} f(x)\,\psi_\eta \right) \chi_\xi^{(2)}$$

It remains to remark that, as we see above, complexes of type 3.2 b) reduce to complexes of type 3.2 a). Theorem 3.5 is proved. ■

Note, that this reduction to curves on L_ℓ permits us to describe all (not only in generic position admissible complexes), consisting of quadrics, passing through the line ℓ, because they transforms to admissible, complexes of curves in L_ℓ (and vice versa), which is completely classified by J.N. Bernstein-S.G. Gindikin. Let us state their result in general form.

Namely, consider the tower of monoidal transformations

$$A : B^q \xrightarrow{\sigma^q} B^{q-1} \xrightarrow{\sigma^{q-1}} \dots \xrightarrow{\sigma^1} B^0 \equiv B$$

where $\sigma^i : B^i \to B^{i-1}$ is the monoidal transformations with the center at an irreducible subvariety $Y_{i-1} \hookrightarrow B^{i-1}$. Let $Z_1, \dots, Z_m$ are hypersurfaces in B^q and $\ell_1, \dots, \ell_m$ – some integer numbers. Denote by $\Gamma(B; A; Z_1 \dots, Z_m; \ell_1, \dots, \ell_m)$ the family of curves from Γ', which can be lifted on B^q, intersect preimage of $Y_0, \dots, Y_{q-1}$ and tangent to $Z_1, \dots, Z_m$ of order $\ell_1, \dots, \ell_m$ respectively.

Denote by d_{i-1} the codimension of Y_{i-1} in B.

It is easy to see that the constructed above family of curves depends on

$$\dim \Gamma - \sum_{i=1}^{m} \ell_i - \sum_{j=0}^{q} (d_j - 1)$$

parameters. We will suppose that this number is greater than dim $B - 1$.

THEOREM 3.9. [BG1-2]. *Let Γ' be a complete family of compact nonsingular rational curves on B. Then the family $\Gamma(B; A; Z_1, \dots, Z_m; \ell_1, \dots, \ell_m)$ is admissible and all admissible families in Γ' can be obtained by this construction.* ■

For example, families in generic position can be described in this language as families of type $\Gamma(B; A; Z_1, \dots, Z_m; \ell_1, \dots, \ell_m)$, where $A : B^1 \to B$ is the monoidal transformation with the center in a subvariety $Y \subset B$ of codimension 2 with r_1 irreducible components.

If we apply this theorem to the family of curves $\check{B}_\xi$ on L_ℓ and consider corresponding complexes of quadrics in $\mathbb{C}P^3$ we obtained the complete list of admissible complexes of quadrics, which contain the given line ℓ.

Exercise 3.10. Try to imagine, how it looks in $\mathbb{C}P^3$ complex of quadrics, corresponding to the complex of curves B_ξ on L, defined by the following tower of monoidal transformations:

$$B^3 \xrightarrow{\sigma^3} B^2 \xrightarrow{\sigma^2} B^1 \xrightarrow{\sigma^1} B^0 \equiv L_\ell$$

where σ^1 is the blow up with the center at a curve $Y_1 \subset L$ and $\sigma^2(\sigma^3)$ is the blow up with the center at a curve $Y_2 \subset (\sigma^1)^{-1}(Y_1)$, (respectively $Y_3 \subset (\sigma^2)^{-1}(Y_2)$).

These complexes may be viewed as *limits* of complexes from example 3.1. We will call such complexes as complexes of type I; complexes from examples 3.2 a), b) where quadric Q and conic C may be singular, we will call as complexes of type II; complexes from example 3.3 as complexes of type III; and complexes from example 3.4 and their limits as complexes of type IV.

THEOREM 3.11. *Every codegree 1 complex of quadrics in $\mathbb{C}P^3$ is complex of type I, II, III or IV.*

We will prove this theorem in §4.

Complexes of type IV cannot be represented in form (4). This fact and the admissibility of complexes of type IV will be proved in [Gon3]. Explicit inversion formulas for complexes of type I, II, III can be found in [Gon5].

3. Contact transformation

In this section we define the action of a contact transformation on subvariety. This definition goes back to Sophus Lie-see [Lie 1], [Lie 2]. We prove that every admissible complex of quadrics of type I - II is contactly equivalent to an admissible complex of curves by a canonical way. I hope also that results of this section clarify the geometry of admissible families of quadrics in $\mathbb{C}P^3$.

We will not differ contact transformations from homogeneous symplectic ones.

It is very important to keep in mind the following well-known lemma.

LEMMA 3.12. *Every closed, irreducible, algebraic homogeneous Lagrangian subvariety in T^*X has the form $T^*_Y X$, where Y is a submanifold in X.* ∎

Suppose that the homogeneous symplectic transformation $\varphi : T^*B_1 \to T^*B_2$

be a birational isomorphism. Then according to this lemma $\varphi(T^*_{B_\xi} B_1)$ can be represented in the form $T^*_{\breve{B}_\xi} B_2$. We will say that the family of subvarieties $\{\breve{B}_\xi\}$ in B_2 is obtained from the family $\{B_\xi\}$ in B_1 by the action of the homogeneous symplectic (or, after projectivisation, contact) transformation φ. In this case we will write $\breve{B}_\xi = \varphi(B_\xi)$.

Example 3.13. (Projective duality). Let $B_1 = P^n$, $B_2 = \breve{P}^n$ be the manifold of hyperplanes in P^n and $C \subset P^n \times \breve{P}^n$ is the incidence manifold. Then we have following diagrams

$$\begin{array}{ccc} & C & \\ \swarrow & & \searrow \\ P^n & & \breve{P}^n \end{array} \qquad \begin{array}{ccc} & T^*_C(P^n \times \breve{P}^n) & \\ {}^{\rho_1}\swarrow & & \searrow{}^{\rho_2} \\ T^*P^n & & T^*\breve{P}^n \end{array}$$

Set $\varphi = (-\rho_2) \circ \rho_1^{-1}$. If B_ξ is a k-dimensional plane in P^n, then $\varphi(B_\xi)$ is the projectively dual $(n-k-1)$-dimensional plane in $\breve{P}^n$.

This construction can be generalised as follows. Let C be submanifold in $B_1 \times B_2$ and $\dim B_1 = \dim B_2$.

Consider a double fibration and its symplectisation

$$(5) \qquad \begin{array}{ccc} & C\,. & \\ \swarrow & & \searrow \\ B_1 & & B_2 \end{array} \qquad \begin{array}{ccc} & T^*_C(B_1 \times B_2) & \\ {}^{\rho_1}\swarrow & & \searrow{}^{\rho_2} \\ T^*B_1 & & T^*B_2 \end{array}$$

Then

$$(6) \qquad \varphi_C := (-\rho_2) \circ \rho_1^{-1} : T^*B_1 \to T^*B_2$$

is a (multivalued) rational homogeneous Lagrangian transformation. The converse is also valid:

PROPOSITION 3.14. *Let $\varphi : T^*B_1 \to T^*B_2$ be a homogeneous Lagrangian transformation. Then there is a submanifold $C \subset B_1 \times B_2$ such that φ has the form (6).*

Proof. Consider the graph $\{x, -\varphi(x)\}$ of the map $-\varphi$. It is a homogeneous Lagrangian subvariety in $T^*B_1 \times T^*B_2$. So by lemma 3.12 it is the conormal bundle to a submanifold $C \subset B_1 \times B_2$. ∎

The following lemma permits us to find geometrically the image of a hypersurface under the action of a contact transformation

LEMMA 3.15. *Let M be a hypersurface in B_1 and $B_2(x) = \{y \in B_2 \mid (x, y) \in C\}$. Then $\varphi_C(M)$ is the envelope of the family $\{B_2(x)\}$ where $x \in M$.* ∎

Example 3.16. The contact transformation, which corresponds to the complex of spheres in R^3, tangent to the plane $z = 0$, transfers the family of spheres in R^3 to the family of paraboloides

(7) $$z = \lambda((x - a)^2 + (y - b)^2 + c)$$

(To prove this, apply lemma 3.15). ■

This family of paraboloids can be defined as the family of all quadrics, passing through two (imagine) intersecting lines at infinity. (Note, that this is 1 : 2 mapping). So it is the admissible family of quadrics of type II with degenerate conic C.

Example 3.17. Let

(8) $$\begin{tikzcd}[column sep=small] & C_L \arrow[dl] \arrow[dr] & \\ \mathbb{C}P^3 & & L \end{tikzcd}$$

be the double fibration, corresponding to an admissible complex L of lines in $\mathbb{C}P^3$. Consider its symplectisation

$$\begin{tikzcd}[column sep=small] & T^*_{C_L}(\mathbb{C}P^3 \times L) \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ T^*\mathbb{C}P^3 & & T^*L \end{tikzcd}$$

and corresponding homogeneous symplectic (rational) transformation

$$\varphi_L := (-\rho_2) \circ \rho_1^{-1}$$

If (8) is a complex of lines, which:

I) intersect the line ℓ

II) or are tangent to a quadric Q

III) are tangent to the ruled surface R

then φ_L transfers a quadric B_ξ, which respectively:

I) passes through the line ℓ

II) passes through the plane conic C

or is tangent to quadric Q along a plane conic

III) is tangent to the ruled surface R along a line

to a rational curve $\check{B}_\xi$ on L.

The constructed contact transformation in the case IIb is the famous line-spheric correspondence of Sophus Lie-see [Lie 1], [Lie 2]. Let us describe it in a more invariant manner.

Let Q_3 be a 3-dimensional quadric. Then the family of all lines on Q_3 can be parametrised by $\mathbb{C}P^3$.

The dual family $\{\Gamma_x\}$ (to the family of lines on Q_3) is the linear complex of lines in $\mathbb{C}P^3$. It can be described as follows. Let us represent $\mathbb{C}P^3$ as a projectivisation of the four-dimensional linear space V equipped with a symplectic structure: $\mathbb{C}P^3 = P(V)$. Then there are 3-parametric family of Lagrangian planes in V, which can be parametrised by the quadric Q_3. If we projectivise these planes, we obtain a linear complex of lines in $P(V)$.

So we obtained the double fibration

$$(9) \qquad \begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ Q_3 & & P^3(V) \end{array}$$

It is clear from proposition 3.9, that it has the degree 2 and the codegree 1.

The group $SO(5)$ acts naturally on the left side of (9) and the group $SP(4)/\{-1\}$ on the right side. So $SO(5) \cong Sp(4)/-\{1\}$ is the symmetry group of the double fibration (9).

The projection of $Q_3 \hookrightarrow \mathbb{C}P^4$ with the center at $x \in Q_3$ transforms complex of lines on Q_3 to complex of lines in $\mathbb{C}P^3$, intersecting the plane conic C, which is the image of the cone of lines on Q_3, passing through x.

Let $\varphi_{Q_3} : T^*Q_3 \to T^*P^3(V)$ be the homogeneous symplectic transformation, connected with the double fibration (9). Let S be a sphere in Q_3 (i.e. hyperplane section of Q_3).. Denote by $\ell_1(S)$ and $\ell_2(S)$ curves in $P^3(V)$, which parametrised two families of lines in S.

LEMMA 3.17. *(Due to Sophus Lie* [Lie 1] [Lie 2]) *$\ell_1(S)$ and $\ell_2(S)$ are lines in $P^3(V)$ such that corresponding planes in V are orthogonal with respect to symplectic structure on V.* ∎

This lemma explained the name "line-spheric" for the transformation φ_{Q_3}.

The linear complex of lines in $P^3(V)$ permits us to define an involutive transformation of subvarieties in $P^3(V)$. Namely, consider the diagram

$$\begin{array}{ccc} & T^*_A(Q_3 \times P^3(V)) & \\ {}^{\rho_1}\swarrow & & \searrow^{\rho_2} \\ T^*Q_3 & & T^*P^3(V) \end{array}$$

Recall, that $\deg \rho_1 = 2$; $\deg \rho_2 = 1$. Let us identify generic parts of $T^*_A(Q_3 \times P^3(V))$ and $T^*P^3(V)$ by the map ρ_2. Then the involution on the total space $T^*_A(Q_3 \times P^3(V))$ of the $2:1$ covering ρ_1 induces the transformation

$$\psi : T^*P^3(V) \to T^*P^3(V); \quad \psi^2 = id$$

For example, if $x \in P^3(V)$ and h_x is the plane in $P^3(V)$ consisting of all lines of the linear complex, passing through x, then $\psi(T^*_xP^3(V)) = T^*_{h_x}P^3(V))$.

Note that the 3-dimensional subspace in V, which corresponds to h_x, is orthogonal, with respect to symplectic structure in V, to the line, corresponding to x.

The next example:

$$\psi(T^*_{\mathfrak{L}(S)} P^3(V)) = T^*_{\mathfrak{L}(S)} P^3(V)$$

If X is a generic surface in $P^3(V)$, then we can define the surface $\psi(x)$:

$$\psi(T^*_X P^3(V)) = T^*_{\psi(X)} P^3(V)$$

It follows from lemma 3.15, that $\psi(X)$ is the envelope of the (2-parametric) family of planes $h_y \subset P^3(V)$, where $x \in h_y$.

Finally, let me recall that horocycles for $SL_2(\mathbb{C})$ are the subsets $g_1 \cdot N \cdot g_2$, where $g_1, g_2 \in SL_2(\mathbb{C})$.

If we identified $SL_2(\mathbb{C})$ with a quadric $ad - bc = 1$, then it can be viewed as an open part of Q_3, and the complex of horocycles transfers to the complex of lines on Q_3.

The explicit local inversion formula for complex of horocycles for $SL_2(\mathbb{C})$ was obtained by I.M. Gelfand and M.A. Naimark in 1947. It plays the crucial role in the problem of finding the explicit Plancherel formula for $SL_2(\mathbb{C})$ (see [GGV]).

§4. PROOF OF THEOREM 3.11

1. Let Γ be a family of submanifolds B_ξ in B. Then a vector $v \in T_\xi\Gamma$ defines the section $\gamma_v(\lambda)$ of the normal bundle to $T^*_{B_\xi}B$ in T^*B. Denote by α the canonical 1-form on T^*B. Then formula

$$\lambda \mapsto -\alpha(\gamma_v(\lambda))$$

defines the map

$$\nu_\xi : T^*_{B_\xi}B \to T^*_\xi\Gamma$$

Denote by $T^*_{y,Y}X$ the fiber of the bundle $T^*_Y X$ at the point y.

LEMMA 4.1. *Let us suppose that* $\pi_2 : A \to \Gamma$ *is a submersion. Then we have the isomorphism*

(10) $$\rho_B : T^*_{a,A}(B \times \Gamma) \to T^*_{x,B_\xi}B \qquad a = (x, \xi)$$

Proof. If $a = (x, \xi)$, then $T_a A \subset T_x B \oplus T_\xi \Gamma$. The mapping $d_a\pi_2 : T_a A \to T_\xi\Gamma$ is epimorphism, so $\operatorname{Ker} d_a\pi_2 = T_x B_\xi \oplus 0$. ∎

So there is the following diagram

$$\begin{tikzcd}[column sep=small] & T^*_A(B\times\Gamma) \arrow[dl, "\rho_B"'] \arrow[dr, "\rho_\Gamma"] & \\ \bigcup_{\xi\in\Gamma} T^*_{B_\xi}B \arrow[rr, "\nu=\bigcup_\xi \nu_\xi"'] & & T^*\Gamma \end{tikzcd} \tag{11}$$

LEMMA 4.2. *Diagram (11) is commutative.*

Proof. It sufficient to check, that if $\lambda \in T^*_{x,B_\xi}B$, then covector $(\lambda, \nu_\xi(\lambda))$ is vanishes on any vector $(v_1, v_2) \in T_a A$ i.e. $\langle \nu_\xi(\lambda), v_2\rangle = -\langle\lambda, v_1\rangle$. But this is the definition of ν_ξ, because $\alpha = pdq$. ■

COROLLARY 4.3. *Suppose that* $\dim B = \dim \Gamma$. *Then the codegree of the double fibration (1) is equal to the degree of the projectivisation of the map*

$$\nu_\xi : T^*_{B_\xi}B \setminus 0 \to T^*_\xi\Gamma \setminus 0$$ ■

Let us denote by $\tilde\Sigma$ the subvariety $\rho_\Gamma(T^*_A(B\times\Gamma)) \subset T$

LEMMA 4.4. [GS1-2] a) $\tilde\Sigma$ *is coisotropic.*

b) Suppose that the map ρ_Γ *is inclusion in the generic point, so we can identify generic parts of* $\rho_\Gamma(T^*_A(B\times\Gamma))$ *and* $\tilde\Sigma$. *Then* ρ_B *coincides with the null-foliation on the coisotropic manifold* $\tilde\Sigma$. ■

2. Let Γ' be the family of all quadrics Q_ξ in $\mathbb{C}P^3$. Then

$$\nu'_\xi : T^*_{Q_\xi}\mathbb{C}P^3 \to T^*_\xi\Gamma' \tag{13}$$

Projectivising (13), we obtain the embedding of Q_ξ in $\mathbb{C}P^8$ provided by the invertible sheaf $O(2)/Q_\xi$ (i.e. $P\nu'^*_\xi\, O(1) = O(2)/Q_\xi$). So $P\nu'_\xi(Q_\xi)$ is the submanifold of degree 8 in $PT^*_\xi\ \Gamma$

Let Γ be a complex of quadrics in $\mathbb{C}P^3$. Then there is the following commutative diagram

$$\begin{tikzcd} & T^*_\xi\Gamma' \arrow[d, "i^*_\xi"] \\ T^*_{Q_\xi}\mathbb{C}P^3 \arrow[ur, "\nu'_\xi"] \arrow[r, "\nu_\xi"'] & T^*_\xi\Gamma \end{tikzcd} \qquad i_\xi : T_\xi\Gamma \hookrightarrow T_\xi\Gamma'$$

Let us projectivise this diagram:

$$\begin{tikzcd} & PT^*_\xi\Gamma' \arrow[d] \\ Q_\xi \arrow[ur, "P\nu'_\xi"] \arrow[r, "P\nu_\xi"'] & PT^*_\xi\Gamma \end{tikzcd} \tag{14}$$

So $P\nu_\xi$ is a birational isomorphism of the quadric $Q_\xi = P^1 \times P^1$ on P^2, provided by a linear subsystem L of the complete linear system $O(2, 2)$ on Q_ξ. Let us classify such linear systems.

I. Suppose that the fixed set of L contain a divisor $\mathcal{D}$. There are 3 possibilities:

a) $\mathscr{D}$ is a line.

b) $\mathscr{D}$ is a conic, or 2 intersecting lines.

c) $\mathscr{D}$ is a pair of non-intersecting lines.

Let us denote by $\mathscr{D}_\xi$ a divisor on Q_ξ which corresponds to $\mathscr{D}$ after identification $P\Sigma_\xi$ and Q_ξ by the map $P\nu'_\xi$ (see (11)).

By lemma 4.4 there is the null-foliation $\rho : \tilde{\Sigma} \to T^*\mathbb{C}P^3$. Then $\rho(\tilde{\Sigma} \cap T^*_\Gamma\Gamma')$ is an isotropic subvariety in $T^*\mathbb{C}P^3$, because $\tilde{\Sigma} \cap T^*_\Gamma \Gamma'$ is an isotropic subvariety in $T^*\Gamma'$.

Denote by π the projection of $T^*\mathbb{C}P^3$ on $\mathbb{C}P^3$. Then

$$Cr := \pi(\tilde{\Sigma} \cap T^*_\Gamma \Gamma')$$

is a subvariety of positive codimension in $\mathbb{C}P^3$.

It follows from results of section 1, that Cr coincides with the critical variety for the family Γ.

Note that

$$\rho(\mathscr{D}_\xi) \subset \rho(\Sigma_\xi) \text{ and } \rho(\mathscr{D}_\xi) \subset PT^*_{Cr}\mathbb{C}P^3 \tag{15}$$

Suppose that $\mathscr{D}_\xi$ is a line. Then Cr contain either a line ℓ, or a one-parametric family of lines, i.e. ruled surface R. In the first case Γ is a complex in the family Γ_ℓ of all quadrics, passing through the line ℓ. In the second Γ is a complex in the family Γ_R of all quadrics, which are tangent to R along a line.

In the case b) Cr contain either a plane conic or a surface M such that quadrics from the complex Γ are tangent to M along a plane conic. In the first case Γ is a complex of quadrics, passing through the plane conic.

Note that quadrics, which are tangent to M along a given conic depends on one parameter. So there is a two-parametric family of conics on M. Therefore M is a quadric.

In the case c) L is a linear (sub)system of $O(0, 2)$, so it defines the map of degree zero, and this case does not occur.

II. Now let us consider the case, when the fixed set of the linear system L is a zero-cycle, i.e. $L = |2H - \Sigma m_i x_i|$ where H is the divisor class of a hyperplane section of the quadric $Q_\xi \subset \mathbb{C}P^3$.

PROPOSITION 4.5. *There is such an index j that $m_j \geqslant 2$.*

Proof. Generic divisor of linear system $|2M|$ is elliptic curve. The linear system $|2H - \Sigma m_i x_i|$ defines a birational isomorphism of the quadric Q_ξ on $\mathbb{C}P^2$. So divisors of this linear system are rational curves, because they are preimages of lines in $\mathbb{C}P^2$.

Therefore they are singular curves. By Bertini theorem there is a point x_j from the base locus of the linear system $| 2H - \Sigma m_i x_i |$ such that all these divisors have singularity at x_j. This means that $m_j \geqslant 2$. ∎

So

$$L = | 2H - 2x_1 - \sum_{i \geqslant 2} m_i x_i | = | 2(H - x_1) - \sum_{i \geqslant 2} m_i x_i | \tag{16}$$

Note that the map, given by the linear system $| H - x_1 |$ is the stereographic projection $p_x : Q_\xi \to P^2$ from the point $x \in Q_\xi$

Thus the line system $| 2(H - x_1) |$ transforms Q_ξ to the Veronese surface in P^5.

Therefore if Γ'' is such a family of quadrics that the map $\underline{P}v''_\xi : Q_\xi \to PT^*_\xi \Gamma''$ is provided by the linear system $| 2(H - x_1) |$ on Q_ξ, then it is the family of all quadrics, passing through the point x and tangent to a given plane, containing x (see (15)).

Let Γ be a codegree 1 complex of quadrics in $\mathbb{C}P^3$ such that Pv_ξ is provided by

$$L = | 2(H - x_1) - x_2 - x_3 - x_4 | .$$

Then $P(T^*_{\xi,\Gamma}\Gamma' \cap \Sigma_\xi) = x_1 \cup x_2 \cup x_3 \cup x_4$ and $\rho(\Sigma_\xi)$ contain 3 Lagrangian

variety $T^*_{M_i}\mathbb{C}P^3$ $(i = 1, 2, 3)$.

So Γ is a complex from example 3.4.

The proof of theorem 3.11 is finished. ∎

§5. A «GENERIC» CODEGREE 1 COMPLEX OF K-DIMENSIONAL SUBMANIFOLDS CAN BE REDUCED TO AN ADMISSIBLE COMPLEX OF CURVES

THEOREM 5.1. (*The Main Theorem*). *Let $\{B_\xi\}$ be a family of submanifolds in B. Suppose that submanifolds of the family, which are tangent to r generic hypersurfaces in B form a complex of codegree 1.*

Then if $r \geqslant 4$, there is a contact transformation, which transfers the family B_ξ to an admissible family of curves. ∎

Remark 5.2. It is sufficient to assume in the formulation of theorem 5.1. that submanifolds B_ξ are tangent to m generic subvarieties of dimension not less than codim $B_\xi - 1$.

Remark 5.3. Complexes, described in the formulation of this theorem, depend on $r \geqslant 4$ generic subvarieties in B. The others codegree 1 complexes in the family B_ξ depend on only 3 generic subvariety. So a "typical" codegree 1 complex in B_ξ is contactly equivalent to an admissible complex of curves.

2. The complete proof of these results will be published in [Gon3]. In this section we indicate the main ingradients of the proof.

First of all, it is easy to prove that Σ_ξ does not lie in a hyperplane in $T^*_\xi\Gamma$. Recall, that if X is a subvariety in $\mathbb{C}P^n$, which does not lie in a hyperplane, then

$$\deg X \geqslant \operatorname{codim} X + 1$$

If the assumption of the theorem 5.1 is holds, then we deduce that

$$\deg P\Sigma_\xi = \operatorname{codim} P\Sigma_\xi + 1$$

Further we need the nice classical Enriques theorem ([E], see also [S.-D.]) which gives complete description of subvarieties of minimal possible degree in P^N, which does not lie in a hyperplane.

Let

$$E = \bigoplus_{i=1}^{r} O(d_i),$$

where $d_i \geqslant 0$, is a vector bundle over P^1. We suppose that $d_i > 0$ for some i. Denote by $\check{P}(E)$ the manifold of all hyperplanes in fibres of the bundle E. Then there is the canonical projection

$$\pi : \check{P}(E) \to P^1$$

Let M be the line bundle over $\check{P}(E)$, whose fibre over the point $x \in \check{P}(E)$ is the quotient of the one of E (over x) on the hyperplane, corresponding to x. Then $\pi_* M = E$ and there is the canonical embedding, defined by the line bundle M:

$$\varphi_M : \check{P}(E) \hookrightarrow \check{P}(H^0(\check{P}(E), M))$$

($\varphi_M(x)$ is a hyperplane in $H^0(\check{P}(E), M)$, consisting of sections, which is zero at x). Easy computation with Chern classes shows that

$$\deg \varphi_M(\check{P}(E)) = \operatorname{codim} \varphi_M(\check{P}(E)) + 1$$

Recall, that the image of P^2 in P^5 by a map, provided by the invertible sheaf $O_{P^2}(2)$ is called the Veronese surface. It also has the minimal possible degree The same property has the bundle $O_{P^2}(2) \oplus O^r_{P^2}$ on P^2.

THEOREM 5.4. *(Enriques) Let X be an irreducible variety in P^N which does not lie in a hyperplane and has minimal possible degree* deg X = codim X + 1. Then X is one of the following:

1. P^N
2. *The quadric in* P^N
3. *The Veronese surface in P^5 or a cone over it in P^{5+r} with "vertex" in $(r-1)$-plane.*
4. *The variety $\varphi_M(X)$, defined above.* ∎

The following theorem is based on the main result of Bernstein-Gindikin [BG1-2].

THEOREM 5.5. *Let Γ' be a family of curves covering an open domain in B and* $\dim \Gamma' > \dim B \geqslant 3$. *Then Γ' is admissible if and only if* $\deg P\Sigma_\xi = \operatorname{codim} P\Sigma_{\xi+1} + 1$. *In this case $P\Sigma_\xi \cong \varphi_M(P(E))$ for some M.* ∎

Let us consider the variety $\check{\Sigma}_\xi \subset T_\xi\Gamma$, which is the dual variety to $\Sigma_\xi \subset T^*_\xi\Gamma$. Recall that $T_\xi\Gamma \cong H^0(P^1, E)$. There is the canonical map

$$H^0(P^1, E \otimes O(-1) \otimes H^0(P^1, O(1)) \to H^0(P^1, E) \tag{18}$$

LEMMA 5.6. *$\check{\Sigma}_\xi \subset H^0(P^1, E)$ can be naturally identified with the image of the map* (18). ∎

dim $H^0(P^1, O(1)) = 2$, so there is a 1-paremetric family of subspaces of codimension r in $T_\xi\Gamma$. We will call it α-subspace.

THEOREM 5.7. *Suppose that Σ is a coisotropic homogeneous subvariety in $T^*\Gamma$ and $P\Sigma_\xi = \varphi_M(P(E))$ where $r \geqslant 4$. Then there is a manifold B and an admissible family of curves on B, parametrised by Γ, such that Σ coincides with the coisotropic subvariety in $T^*\Gamma$, defined by this family.* ∎

So, there is a desired birational homogeneous symplectomorphism $\varphi : T^*B \to T^*\widetilde{B}$, such that the following diagram

$$\begin{tikzcd} & \Sigma \arrow[dl, "\rho_B"'] \arrow[dr, "\rho_{\widetilde{B}}"] & \\ T^*B \arrow[rr, leftrightarrow, "\varphi"'] & & T^*\widetilde{B} \end{tikzcd}$$

is commutative, where ρ_B and $\rho_{\widetilde{B}}$ are the null-foliations for Σ.

The manifold $\widetilde{B}$ constructed as follows. We prove that assumption in theorem 5.7 implies that for every α-subspace there is the unique submanifold $Y \subset \Gamma$ such that the tangent space at every point $y \in Y$ is an α-subspace. We will call such submanifolds in Γ as α-submanifold. Then $\widetilde{B}$ parametrise α-submanifold in Γ.

The *existence* of α-submanifolds in Γ is the most complicated part of the proof. It based on main results of [Gon4].

I think, that subvarieties of minimal possible degree in P^n play a key role in integral geometry. Let me formulate 2 results, illustrating this idea. (The proofs will be punblished in [Gon 5]).

THEOREM 5.8. Let X be a subvariety in P^n, which does not lie in a hyperplane. Then the set of hyperplane sections of X, which are tangent to codim X, generic algebraic hypersurfaces in X is admissible if and only if $\deg X = \operatorname{codim} X + 1$.

EXAMPLE 5.9. Admissible complexes of quadrics of type I, II, IV are birationally isomorphic to the family of hyperplane sections of:

I. $\varphi_M(\check{P}(E))$, where $E = O \oplus O(2) \oplus O(2)$

II. 3-dimensional quadric in P^4

IV. The cone (in P^6) over the Veronese surface in P^5.

Namely, if we removed the tangencv conditions from the definitions of these complexes, we obtain linear systems, provided desired birational isomorphisms.

Finally, *every* admissible family of curves on an *algebraic* surface can be *canonically* realised by hyperplaine sections of a surface of minimal possible degree in $\mathbb{C}P^n$:

THEOREM 5.9. Let $X^2 \subset \mathbb{C}P^n$ be one of the following surfaces:

$\mathbb{C}P^2$ $(n = 2)$, the Veronese surface in $\mathbb{C}P^5$ or a scroll $\varphi_M(\check{P}(E)) \subset \mathbb{C}P^{k_1+k_2+1}$, where $E = O(k_1) \oplus O(k_2)$.

Then

a) The family of hyperplane sections of X^2, tangent to given algebraic curves $M_1, \dots, M_k \subset X^2$ of order $l_1, \dots, l_k$, $(l_1 + \dots + l_k \leqslant n - 2)$ is admissible.

b) Every admissible family of curves on an algebraic surface is birationally isomorphic to just one of these families. ∎

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